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Causal relations via linking in twistor space

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Abstract

We investigate the relationship between two different notions of linking, one homological and one holomorphic, in a twistorial context. The two notions turn out to be surprisingly closely related. © 1998 Elsevier Science B.V.

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1. Homological linking and causality

Causal relations are extremely easy to describe in Minkoswki space. If we make a choice of a Minkowskian coordinate system q^a, $a = 0, \ldots, 3$, so that the metric is

$$\mathrm{d}s^2 = (\mathrm{d}q^0)^2 - \sum_{i=1}^{3}(\mathrm{d}q^i)^2$$

then two points x and y with coordinates q_x^a and q_y^a are *causally separated* if and only if

$$(q_x^0 - q_y^0)^2 \geq \sum_{i=1}^{3}(q_x^i - q_y^i)^2$$

and x is to the future (past) of y if $q_x^0 > q_y^0$ ($q_x^0 < q_y^0$). The space–time point x is said to be *chronologically separated* from y if the former inequality is strict. If x is chronologically

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separated from y, we write $x \in I(y)$; if x is to the future (past) of y, then $x \in I^+(y)$ ($x \in I^-(y)$). We can also understand this relationship in a much more geometric way, though.

For choose a Cauchy surface $\mathcal{S}$ containing x. Then the set of all light rays passing through y will intersect $\mathcal{S}$ in a two-sphere $x_{\mathcal{S}}$. We now find that x and y are causally separated if x lies inside or on $y_{\mathcal{S}}$, and chronologically separated if x lies inside it. Furthermore, we can associate an orientation to the sphere: for the future pointing tangents to the light rays through y will point into $y_{\mathcal{S}}$ if y is to the future of x, and outward if y lies to the past.

Using this, we can give an explicit definition of the winding number of $y_{\mathcal{S}}$ around x in $\mathcal{S}$. For the fundamental class of $y_{\mathcal{S}}$ is mapped into either 1 or -1 in $H_2(\mathcal{S} \setminus \{x\})$; we define the winding number of $y_{\mathcal{S}}$ round x to be this value. By an appropriate choice of orientation on $\mathcal{S}$, we can ensure that y is to the chronological future of x if $y_{\mathcal{S}}$ has a winding number of 1 round x, and to the chronological past if the winding number is -1.

There is yet another picture of this relationship, which we can obtain by considering $\mathbb{PN}^I$, the space of all null geodesics in Minkowski space. We can represent x and y in $\mathbb{PN}^I$ by X and Y, the set of all null geodesics passing through x and y, respectively. Then $\mathbb{PN}^I \cong \mathbb{R}^3 \times S^2$, and $\mathbb{PN}^I \setminus X \cong \mathbb{R} \times S^2 \times S^2$, so that $H_2(\mathbb{PN}^I) = \mathbb{Z}$, and $H_2(\mathbb{PN}^I \setminus X) = \mathbb{Z} \oplus \mathbb{Z}$. To find a homological definition of the linking number of X and Y in $\mathbb{PN}^I$, we need (intuitively) to find the image of the fundamental class of Y in $H_2(\mathbb{PN}^I \setminus X)/H_2(\mathbb{PN}^I)$ (cf. the definition of linking number in [11]). In general, $H_n(\mathbb{R}^{n+1} \times S^n)$ is a quotient group, rather than a subgroup of $H_n(\mathbb{R} \times S^n \times S^n)$. However, it is possible to identify $H_2(\mathbb{R}^3 \times S^2)$ with a subgroup of $H_2(\mathbb{R} \times S^2 \times S^2)$: one way of doing this is to embed $\mathbb{PN}^I$ in $\mathbb{R}^5$, and then observe that the linking number of the images of X and Y is independent of the choice of embedding, as set out in [8]. This enables one to identify $H_2(\mathbb{PN}^I)$ as a subgroup of $H_2(\mathbb{PN}^I \setminus X)$, and so define the linking number of Y round X, denoted $L(X, Y)$, as the image of the fundamental class of Y in the quotient group.

The integer so-obtained is equal to the winding number of $y_{\mathcal{S}}$ about x in $\mathcal{S}$. Thus, we can define a linking number for two spheres each corresponding to the set of all null geodesics through a point of Minkowski space represented in $\mathbb{PN}^I$, and this linking number exactly captures the chronology relationship. In fact, by an appropriate choice of orientations, we have $L(X, Y) = 1$ if and only if $y \in I^+(x)$, and $L(X, Y) = -1$ if and only if $y \in I^-(x)$.

2. Holomorphic linking

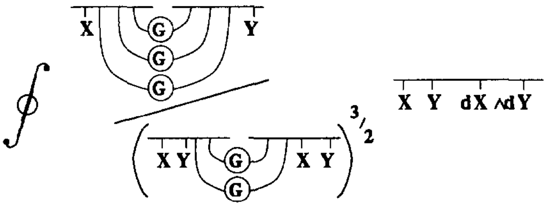
The above notion of linking is as close as we can get to that of two S^1's in $\mathbb{R}^3$, where if C_1 and C_2 are two oriented S^1's, we can obtain the linking number $L(C_1, C_2)$ of C_1 round C_2 by finding the image of the fundamental class of C_1 in $\mathbb{R}^3 \setminus C_2$. Unfortunately, the analogy is weaker than one might prefer, essentially because the second homology group of $\mathbb{PN}^I$ is non-trivial even before L_P is removed.

However, there are other ways of finding $L(C_1, C_2)$. One can draw a projection of the link to $\mathbb{R}^2$, and count crossings with the appropriate sign [6]. This admits of no obvious analogue.

More intriguingly, one can use the following integral, as described in [4], where $\mathbf{x}$ and $\mathbf{y}$ range over C_1 and C_2, respectively:

$$L(C_1, C_2) = \frac{1}{4\pi} \oint \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathrm{d}\mathbf{x} \wedge \mathrm{d}\mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3}.$$

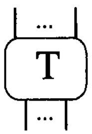
As pointed out in [9] one can write this integral projectively and since the integrand is analytic, allow the variables to become complex. This yields the following expression for L:



(1)

where the integral is carried out over a one real dimensional contour in each of the one complex dimensional complex curves X and Y, and G is some symmetric holomorphic metric on $\mathbb{PT}$. The diagrammatic notation of Penrose [10] is used here for clarity. This may be summarized by the property that for any manifold $\mathcal{M}$ a tensor field $T^{a_1,\dots,a_r}{}_{b_1,\dots,b_s}$ on $\mathcal{M}$ of type (r, s), i.e. with r contravariant and s covariant indices, is represented diagrammatically in the form

r lines



s lines

and that lines joining index pairs denote contraction according to the Einstein summation convention. A horizontal bar is taken to represent the alternating symbol, or totally skew tensor, on $\mathcal{M}^n$, more usually written as $\epsilon_{a_1,\dots,a_n}$.

Eq. (1) gives us a quantity defined for two holomorphic complex curves in $\mathbb{PT}$ that bears a strong formal resemblance to a linking number. (Observe however that it is skew-symmetric in X and Y, while the corresponding integral for real curves in $\mathbb{R}^3$ is symmetric.) The holomorphic link integral is invariant under small holomorphic changes of the contours and small holomorphic variations in G provided that the integrand remains non-singular [9]. We shall call this quantity the *holomorphic linking number* of X and Y and denote it by $L_{\mathcal{H}}(X, Y)$.

The question we pose is what relation may exist between $L_{\mathcal{H}}$ and the homological linking number described in Section 1. The surprising result is that the holomorphic linking number is actually closely related to the homological linking number defined for spheres within $\mathbb{PN}^I$.

3. Holomorphic linking and the quantum Kählerian structure

The actual evaluation of the holomorphic linking number involves some fairly subtle problems. However, following a suggestion of Hodges, reported in [9], on how to represent the holomorphic linking number in terms of twistor diagrams, one can find a cohomological interpretation of the quantity, and so re-express the integral in terms of certain fields on space–time.

For let ϕ and ψ be fields on (complexified) Minkowski space. Such fields may be split up into positive and negative frequency parts [1] so that we may regard ϕ as the pair (ϕ^+, ϕ^-). There is a triple of natural structures available, a *quantum Kählerian structure*, namely a positive definite metric g, a symplectic structure Ω, and a complex structure J, satisfying the compatibility relationship

$$g(\phi, \psi) = \Omega(\phi, J\psi). \tag{2}$$

If we take g to be the Hilbert space inner product on our fields, and J to be defined by

$$J\phi = \mathrm{i}(\phi^+ - \phi^-),$$

then we may regard Ω as defined by the above consistency relation [2,7]. Explicitly the symplectic structure is represented on space–time as

$$\Omega(\phi, \psi) = \int_{\Sigma} (\phi \nabla_a \psi - \psi \nabla_a \phi)\, \mathrm{d}^3 \Sigma^a$$

for an arbitrary spacelike cross section of Minkowski space. The integral is independent of the choice of Σ whenever ϕ, ψ satisfy the wave equation. The action of the complex structure J on a space–time field or potential φ (with any index structure) satisfying $\nabla^2 \varphi = 0$ throughout Minkowski space, whose positive and negative frequency parts satisfy certain decay requirements (see [2] for details), is given by the three-surface integral

$$J[\varphi](x) = -\frac{1}{2\pi^2} \int_{\Sigma(x)} \frac{1}{K^2(x, x')} (\overleftarrow{\nabla}'^a - \overrightarrow{\nabla}'^a) \varphi(x')\, \mathrm{d}^3 \Sigma'_a,$$

where $K^2 = (x^a - x'^a)^2$, Σ_a is future pointing, and $\Sigma(x)$ is a once differentiable real spacelike hypersurface constrained to contain the point of evaluation x. Recasting these results in twistor form it can be shown that if ϕ and ψ are defined by the holomorphic curves $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{PT}$ respectively, then there exists a relation between the holomorphic linking number of these curves and the metric and symplectic products of the associated fields. We have the following result.

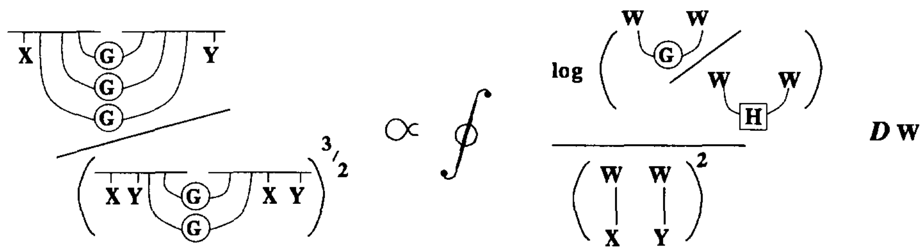
Proposition. *If ϕ and ψ are defined by the holomorphic curves $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{PT}$, respectively, then the relation*

$$L_{\mathcal{H}}(\mathcal{X}, \mathcal{Y}) = \frac{g(\phi, \psi)}{\Omega(\phi, \psi)} \tag{3}$$

holds, whenever one of $\mathcal{X}$ or $\mathcal{Y}$ lies in $\mathbb{PT}^+$ and the other lies in $\mathbb{PT}^-$.

Proof. We outline the proof of this result in two main steps. Firstly, as pointed out in [9], one can rewrite the holomorphic link integral in twistor form. We have the following result.

Lemma. *For an appropriate choice of contour*



(4)

in which the square symbol containing H represents some metric distinct from G in general position.

Proof of the lemma. This result first appears in [9], although no specification there of the required contour is given. What is needed is the *Pochammer contour* [12] which we shall denote $\mathcal{P}$, and the details of how to apply this to (4) appear in Section 8.3 of [2]. In brief the construction is as follows. A useful preliminary observation, due to Hodges, is that

$$\oint \frac{\log[G(W,W)/H(W,W)]}{(W\cdot X)^2(W\cdot Y)^2}\mathcal{D}W \equiv G(\partial_X,\partial_Y)\oint \frac{\log[G(W,W)/H(W,W)]}{(W\cdot X)(W\cdot Y)G(W,W)}\mathcal{D}W. \tag{5}$$

A priori the integral on the right-hand side above depends on both metrics G and H, and it is in the specification of the contour of integration that the dependence on H disappears. The topology of the required contour is $S^1\times S^1\times\mathcal{P}$. The two S^1 components encircle the simple poles situated where $W\cdot X$ and $W\cdot Y$ vanish. The effect of the $S^1\times S^1$ integration (cf. [10]) is to restrict the twistor integral to a pure spinor integral, in which the spinor variable serves as a coordinate on the line joining X and Y in $\mathbb{PT}$. Explicitly the twistor integral reduces to

$$\oint \frac{\log[(G_{AB}\eta^A\eta^B)/(H_{AB}\eta^A\eta^B)]}{G_{AB}\eta^A\eta^B}\Delta W \tag{6}$$

with

$$W_\alpha=(\eta_A,\sigma^{A'}),\qquad \Delta W=\eta_A\,\mathrm{d}\eta^A,$$

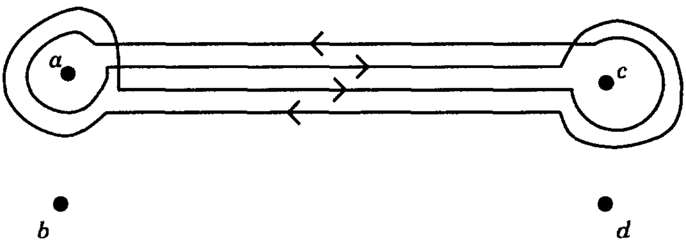
where $G_{AB}=G_{(AB)}$ and $H_{AB}=H_{(AB)}$ are the primary spinor parts [10] of the twistor metrics $G_{\alpha\beta}$ and $H_{\alpha\beta}$ respectively. We can rescale the twistor W_α so that

$$G_{AB}\eta^A\eta^B=(z-a)(z-c),\qquad H_{AB}\eta^A\eta^B=(z-b)(z-d),\quad \eta^A=(z,1)$$

and then (6) becomes

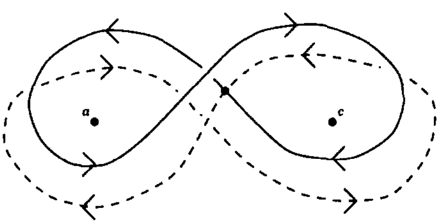
$$\oint \frac{1}{(z-a)(z-c)} \log\left[\frac{(z-a)(z-c)}{(z-b)(z-d)}\right] \mathrm{d}z \tag{7}$$

in the complex plane of the variable z. To yield independence of (4) on $H_{\alpha\beta}$ we seek a contour for (7) above such that the result is independent of b and d. A detailed investigation reveals that a Pochammer contour is required, as we illustrate below.



The Pochammer contour.

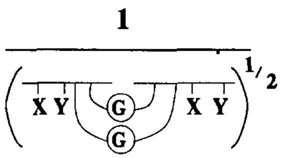
The more usual application of this Pochammer contour encountered in the literature is for an integrand containing an expression of the form $(a-z)^{\lambda}(c-z)^{\mu}$ where λ, μ are complex powers (as occurs in the integral representation of the *beta function*, cf. [12]). The Pochammer contour can be seen to consist of two figures of eight



In the case of the integrand $(a-z)^{\lambda}(c-z)^{\mu}$, traversal of a single constituent figure of eight takes one onto a different sheet of its Riemann surface, and only traversal of both figures yields a closed contour. The situation as regards (7) is different however, in that traversal of a single figure of eight yields a closed contour, and the two figures by themselves are *period contours* for (7). Integration over each period separately yields a result that depends on b and d, and thus on the twistor metric $H_{\alpha\beta}$. When both periods are traversed however, and thus the integration is over $\mathcal{P}$, the dependence on H is eliminated. The result of this integration is non-zero since the two periods cannot be deformed into one another on the Riemann surface of the integrand (7). Calculation yields the result of the integration to be

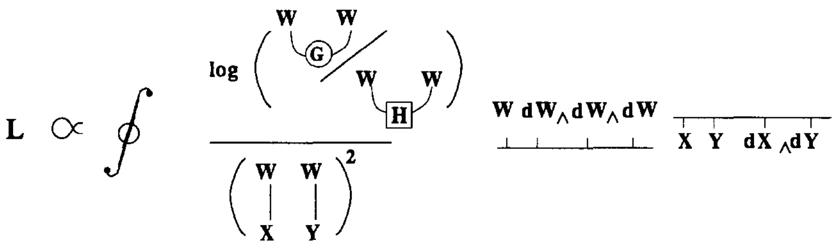
$$\frac{1}{c-a} = \frac{1}{(G^A{}_B G_A{}^B)^{1/2}}$$

and the right-hand side above can be written in twistor terms diagrammatically as



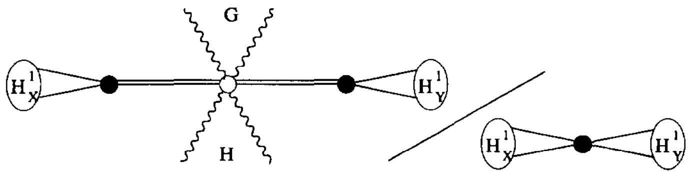
The action of $G(\partial_X, \partial_Y)$ on this produces the required result. □

Using the above lemma we see that the link number is given by



(8)

As pointed out in [9] the corresponding twistor diagram expression for L is

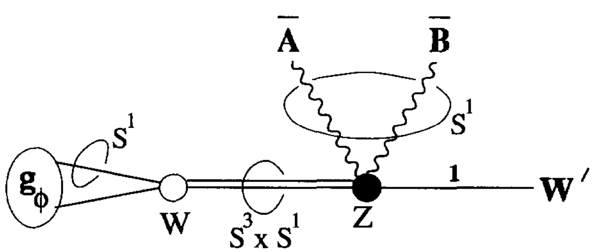


(9)

Here $H^1_{\mathcal{X},\mathcal{Y}}$ are elements of the cohomology group $H^1(\mathbb{PT}, \mathrm{O}(-2))$ based on the holomorphic curves $\mathcal{X}$ and $\mathcal{Y}$, respectively. That is,

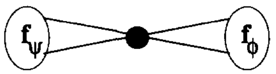
$$\mathcal{X} \longleftrightarrow H^1_{ij}(\mathcal{X}) \equiv p_{[i}q_{j]} =: (p \bullet q)_{ij}$$

($H^1(\mathcal{X})$ is said to be the *dot product* of p with q) for holomorphic functions $p, q \in H^0(\mathbb{PT})$ whose common singularity region defines the holomorphic curve $\mathcal{X}$. A similar construction holds for $\mathcal{Y}$. It has been shown (Eq. 7.2.60 in [2]) that the twistor representation of the complex structure J corresponding to a mixed frequency twistor function $g_\phi(W)$ for a spacetime field ϕ, is given by a derived twistor function $g_{J[\phi]}(W')$ with diagram



(10)

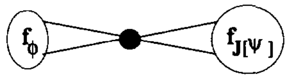
It will be understood in this expression and in the following, that the twistor functions are defined as elements of first cohomology on a complex thickening of $\mathbb{PN}$. This provides the natural description of mixed frequency fields in Čech cohomological terms. For a more detailed account of these matters see [2,5]. We have the result (cf. Lemma 7.2 in [2]) that the symplectic product between a pair of mixed frequency fields is given by a twistor integral with diagram



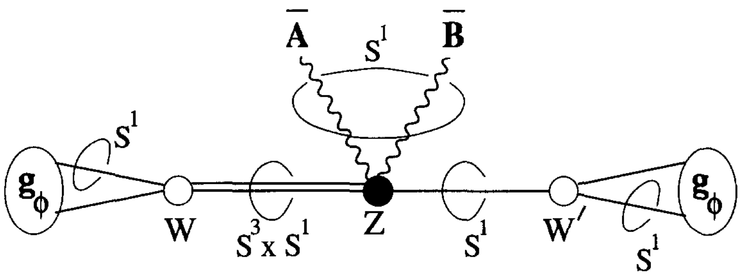
$$\equiv \quad \Omega(\Psi, \phi)$$

Thus the corresponding metric product $g(\phi, \psi)$, which by (2) is given by

$$g(\phi, \Psi) =$$



may be expanded using (10) to give



Deformation of the quadric boundary $H(W, W) = 0$ in the numerator of (9) to become a pair of planes completes the proof of the proposition. □

Remarks. The result tells us that two complex lines in $\mathbb{PT}$ each of which lies in either $\mathbb{PT}^+$ or $\mathbb{PT}^-$ must have a holomorphic linking number of ± 1 if exactly one lies in each region. The cohomological properties of the twistor diagrams also ensure that the holomorphic linking number is not defined if either of $\mathcal{X}$ or $\mathcal{Y}$ meets $\mathbb{PN}$, and is zero if both lie in $\mathbb{PT}^+$ or $\mathbb{PT}^-$ [2].

Now we can choose $\mathcal{X}$ and $\mathcal{Y}$ to be complex lines in $\mathbb{PT}^+$, $\mathbb{PT}^-$ respectively, and, in particular, such that $\mathcal{X}$ corresponds to a point $-ix^a$, with x^a a point in the chronological future of 0 in real Minkowski space; and similarly such that $\mathcal{Y}$ corresponds to a point $-iy^a$ with y^a a point in the chronological past of 0 in real Minkowski space. Furthermore, the motion $x^a \to -ix^a$ in complex Minkowski space induces the mapping $(\omega^A, \pi_{A'}) \to (-i\omega^A, \pi_{A'})$ in $\mathbb{PT}$. If we denote this action by j, then $\mathcal{X} = j(X)$, $\mathcal{Y} = j(Y)$, and we have

$$L(X, Y) = L_{\mathcal{H}}(j(X), j(Y)).$$

4. Issues of conformal and holomorphic invariance

At first sight, a holomorphic linking expression for causal relations in Minkowski space seems implausible from the point of view of conformal invariance. We must understand how an integral within $\mathbb{PT}$, which is holomorphically invariant, can give a result that is not conformally invariant in complexified compactified Minkowski space. Indeed, since S^2's do not link in six dimensions, it might seem that the holomorphic linking number could not calculate anything related to homological linking in $\mathbb{PN}^I$.

In fact, there was an undeclared subtlety in Section 3. Although one can use the holomorphic linking number in $\mathbb{PT}$ to calculate the linking number in $\mathbb{PN}^I$, the above procedure only gives the correct answer for chronologically separated points if one is in the chronological future of the *origin* in Minkowski space while the other is in the chronological past. For if x and y are timelike separated, but both lie to the future of the origin, then both X and Y will be carried into $\mathbb{PT}^-$, and $L_{\mathcal{H}}(j(X), j(Y))$ will be zero.

To carry out the required procedure, we have to choose the origin of Minkowski space to be the point midway between x and y. Once the infinity twistor is chosen, i.e. the line I in $\mathbb{PT}$ [10], we have complexified Minkowski space as an affine space: the midpoint of the line joining x and y is then well-defined. Choosing that as the origin of Minkowski space, we can then apply the above operation. Once this choice is made, $L_{\mathcal{H}}(j(X), j(Y))$ calculates the linking number.

Explicitly, any space–time point $x^a \in \mathbb{CM}^\sharp$ corresponds to a simple skew twistor $X^{\alpha\beta}$, with the normalization that for points in finite Minkowski space the relation $X^{\alpha\beta}I_{\alpha\beta} = 2$ holds, where $I^{\alpha\beta}$ is the infinity twistor (this convention is in accordance with [10]). Let us also fix $O^{\alpha\beta}$ to be the simple skew twistor representing the line in twistor space that corresponds to some chosen origin in Minkowski space. Then the following relations hold:

$$X^{\alpha\beta}O_{\alpha\beta} = -(x_a x^a), \qquad X^{\alpha\beta}Y_{\alpha\beta} = -(x^a - y^a)(x_a - y_a).$$

For two points x^a, y^a in *affine* Minkowski space represented by simple skew twistors $X^{\alpha\beta}$ and $Y^{\alpha\beta}$, respectively, the point $\frac{1}{2}\mathrm{i}(x-y)^a$ has twistor representation

$$\begin{aligned} &\tfrac{1}{2}\mathrm{i}(x-y)^a \longleftrightarrow \tfrac{1}{2}\mathrm{i}(X^{\alpha\beta} - Y^{\alpha\beta}) + f(X,Y)I^{\alpha\beta} + O^{\alpha\beta}, \\ &f(X,Y) := \tfrac{1}{4}(\mathrm{i}[Y^{\alpha\beta} - X^{\alpha\beta}]O_{\alpha\beta} - \tfrac{1}{2}X^{\alpha\beta}Y_{\alpha\beta}). \end{aligned}$$

Thus clearly the operation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}\mathrm{i}(x-y) \\ \frac{1}{2}\mathrm{i}(y-x) \end{pmatrix},$$

in affine $\mathbb{M}$ breaks conformal invariance, via the requirement that the infinity twistor $I^{\alpha\beta}$ be specified in its twistor description.

In the above description, we have seen how to relate a twistor diagram based on two twistor lines in $\mathbb{PN}^I$ by rotating these lines so that if they are causally separated one lies in $\mathbb{PT}^+$ and one lies in $\mathbb{PT}^-$; the diagram is evaluated in terms of cohomology based on $\mathbb{PN}$.

It is also possible to take a slightly different perspective on this. Consider again the rotation $j : \mathbb{PT} \to \mathbb{PT}$, but now, instead of using it to move X and Y, consider $j(\mathbb{PN})$, the image of $\mathbb{PN}$ under this map. If we now reinterpret the twistor diagram (9) as defining cohomology based on a thickening of this surface, rather than on a thickening of $\mathbb{PN}$, then we immediately obtain a holomorphic linking number that agrees with the homological one when X and Y both lie in $\mathbb{PN}^I$. This corresponds to choosing a new metric G on $\mathbb{PT}$; recall that the value of the holomorphic linking number was invariant only under sufficiently small changes in G.

Furthermore, we can now regard the holomorphic linking number as a genuine generalization of the homological one. If X and Y are linked in $\mathbb{PN}^I$ in the homological sense, then we regard X as a fibre of $\mathbb{PN}^I$ regarded as an S^2 bundle over S, and Y can only be deformed (within $\mathbb{PN}^I$) into another fibre of this bundle if at some stage in the deformation Y intersects X, and at this stage the homological linking number is undefined. Although it is possible to deform Y into another fibre of the bundle by moving it out of $\mathbb{PN}^I$, at some stage of this process Y must pass through $j(\mathbb{PN})$, and this will cause the holomorphic linking number to be undefined.

We note finally that it follows that we can use the holomorphic linking number to give a formal generalization of causal structure to complex Minkowski space. However, an interpretation of this in terms of domains of dependence for fields determined by initial data remains to be investigated.

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References

[1] T.N. Bailey, L. Ehrenpreis, R.O. Wells, Jr., Weak solutions of the massless field equations, Proc. Roy. Soc. London A 384 (1982) 403–425.
[2] T.R. Field, The complex quantum structure, D. Phil. thesis, Oxford University, 1997.
[3] T.R. Field, Geometrical aspects of the Kähler structure for the quantum state space of zero rest-mass fields, ESI, Erwin Schrödinger International Institute for Mathematical Physics, Preprint No. 535, 1998; J. Math. Phys. (1998), submitted.
[4] H. Flanders, Differential Forms with Applications to the Physical Sciences, Academic Press, New York 1963.
[5] M.L. Ginsberg, Scattering theory and the geometry of multi-twistor spaces, Trans. Amer. Math. Soc. 276 (2) (1983) 789–815.
[6] L. Kaufmann, On Knots, Princeton University Press, Princeton, NJ, 1987.
[7] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. 2, Wiley Classics Library, 1996.
[8] R. Low, Twistor linking and causal relations Class Quantum Grav 7 (1990) 177–187.
[9] R. Penrose, Topological QFT and Twistors: Holomorphic linking, Twistor Newsletter, Mathematical Institute, Oxford University, 27, 1988, pp. 1–3
[10] R. Penrose, M.A.H. MacCallum, Twistor theory: an approach to the quantization of fields and space–time, Phys. Rep. 6C (1972) 241–315
[11] E.H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1962.
[12] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, 2nd ed., Cambridge University Press, Cambridge, 1915.